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Internal waves in a sheeted thermocline

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The eigenvalue problem for internal waves in a thermocline for which the density profile may exhibit finite discontinuities is formulated as a homogeneous Fredholm integral equation. The corresponding quadratic functional yields both upper and lower bounds for the dominant-mode eigenvalue and lower bounds for the remaining eigenvalues (which are proportional to the square of the wave speed) for any value of the wavenumber. The lower bounds based on simple trial functions appear to provide adequate approximations for typical density profiles and all wavenumbers. The upper bound (which is based on a Schwarz inequality and does not require the choice of a trial function) is sharp only for relatively long waves. A simple approximation is developed for the effect of the free surface on the dominant mode. Two algebraic formulations are given for a thermocline of homogeneous layers separated by a finite number of sheets, across each of which the density is discontinuous. The various approximations are compared with the exact results for a thermocline with a hyperbolic-tangent density profile, a three-sheeted thermocline, and a five-sheeted model of the summer thermocline in the Mediterranean.

1. Introduction

It now appears to be established that typical summer (and perhaps most) thermoclines have a step-like structure, comprising a series of *layers*, in which the temperature changes gradually, separated by thin *sheets*, across which the temperature changes sharply (see, e.g. Woods 1968). The usual formulation of the internal-wave problem, in terms of a differential equation and associated boundary conditions, and especially its solution through the WKB approximation, assumes that the density (or temperature) gradient is a smooth function of depth; accordingly, it is not well suited to such a thermocline. We develop here an integral-equation formulation that leads directly to variational approximations for the wave speeds of the more important modes and is applicable to any density profile that is piecewise continuous.

Let $\rho(y)$ be the density profile, where y is measured positive upwards from an appropriately defined plane in the thermocline. We assume that $\rho'(y) \leqslant 0$ ($\rho' \downarrow -\infty$ in the limiting case of a sheet of infinitesimal thickness) and

$$\rho_+ \leqslant \rho(y) \leqslant \rho_-$$

for all y, and measure the overall strength of the thermocline by

$$\beta = (\rho_- - \rho_+)/(\rho_- + \rho_+) \tag{1.1}$$

and the strength of a sheet with midplane at $y = y_n$ by

$$\beta_n = \{\rho(y_n^-) - \rho(y_n^+)\}/\{\rho(y_n^-) + \rho(y_n^+)\} \equiv \epsilon_n \beta. \tag{1.2}$$

The Boussinesq approximation, which we invoke in (2.5) *et seq.*, implies that

$$\beta \ll 1 \quad \text{and} \quad \sum_n \epsilon_n \leqslant 1, \tag{1.3a,b}$$

where equality applies in (1.3*b*) if and only if (as in § 5) the layers are homogeneous.

Appropriate characteristic lengths are L, a characteristic thickness for the thermocline as a whole, l, a characteristic thickness of the layers, l_s, a characteristic thickness of the sheets, d, the depth of the thermocline below the surface, D, the depth of the ocean, $1/k$, the reciprocal wavenumber. We assume that

$$l_s \ll (1/k, l, L) \ll d \ll D \tag{1.4}$$

except where noted, set $l_s = 0$ and $D = \infty$ throughout the subsequent development, and set $d = \infty$ in the initial formulation of §§ 2 and 3. The assumption $kl_s \ll 1$ excludes those waves associated with the structure of the sheets (these waves may be rendered unstable by the shear induced by the longer waves; see Woods (1968)). Waves for which $l_s \ll 1/k \ll l$ are confined to the neighbourhood of, and determined essentially by the density jump across, each of the individual sheets, as characterized by β_n. Waves for which $1/k$ is in the scale range (l, L) are determined by the thermocline as a whole. Waves for which both $kd \ll 1$ and $kL \ll 1$ may present special difficulties, especially with regard to the Boussinesq approximation (see Long 1965; Benjamin 1967); however, these difficulties do not appear to be significant in the present context.

Let c be the wave speed of an internal wave of infinitesimal amplitude. The equations of motion, together with appropriate boundary conditions, yield a dispersion relation between k and c or, equivalently, between the dimensionless parameters

$$\alpha = kL \quad \text{and} \quad \mu = kc^2/\beta g \equiv 1/\lambda \quad (k > 0) \tag{1.5a,b}$$

for each of a discrete spectrum of modes, say $\mu = \mu_n(\alpha)$, $\mu_1 > \mu_2 > \ldots$ (the assumption that the spectrum is discrete imposes restrictions on $\rho'(y)$ that are trivial in the present context). In defining μ, we refer c to $(\beta g/k)^{\frac{1}{2}}$, the speed of an interfacial wave between two semi-infinite, homogeneous liquids (Lamb 1932, § 231). The subscript zero is appropriately reserved for the free-surface mode, for which $\mu_0 = O(1/\beta)$ as $\beta \downarrow 0$; the Boussinesq approximation, consistently applied, suppresses this mode and replaces the free surface by a rigid boundary.

The dominant mode, $\mu = \mu_1$, is typically the most important for the motion of the thermocline as a whole for small or moderate values of α, although the second mode, $\mu = \mu_2$, could be the most important for certain types of excitation. The aforementioned result for an interfacial wave implies that $c^2 \to \beta g/k$ or, equivalently, $\mu_1 \to 1$ for a thin ($kL \ll 1$), deep ($kd \gg 1$) thermocline. It may be shown (either from Sturm–Liouville theory or from the results in § 3 below) that this limiting result provides an upper bound for μ or, equivalently, a lower bound for λ. Drazin & Howard (1961)† show that the limit is approached according to

$$\lambda_1 = 1+\alpha+O(\alpha^2) \quad (\alpha \equiv kL \downarrow 0,\, kd \gg 1), \tag{1.6a}$$

in which
$$L \equiv 2\int_{-\infty}^{\infty} \frac{(\rho-\rho_+)(\rho_- - \rho)\,dy}{(\rho_- - \rho_+)^2} \tag{1.6b}$$

appears as a suitable measure of the thickness of the thermocline.

The *a priori* restriction $kd \gg 1$ is, of course, untenable for sufficiently long waves in the dominant mode, for which c must exhibit the limiting behaviour (cf. Benjamin 1967)

$$c^2 = c_0^2\{1-kL_1+O(k^2d^2)\} \quad (kd \downarrow 0), \tag{1.7a}$$

where
$$c_0^2 \equiv 2\beta g L_0, \tag{1.7b}$$

and the lengths L_0 and L_1 are $O(d)$. We obtain approximations to L_0 and L_1 for $L/d \ll 1$ in equations (4.7a,b). These approximations suggest that the effects of finite depth on the dominant mode may be determined approximately through the relation

$$c^2(k,L,L_0) \doteqdot c^2(k,L,\infty)\{1-\exp(-2kL_0)\}. \tag{1.8}$$

The remaining μ_n tend to zero with α and are affected only quantitatively, rather than qualitatively, by the free surface. Sturm–Liouville theory implies $\mu_n = O(1/\alpha)$ as $\alpha \uparrow \infty$ for a continuous thermocline, but if the thermocline contains N sheets of strength β_n there will be N eigenvalues, say ${}_1\mu$, ${}_2\mu$, …, ${}_N\mu$, that tend to finite limits. Invoking the (readily confirmed) hypothesis that these sheets move independently for large α, we obtain $c^2 \sim \beta_n g/k$ or, equivalently,

$$_n\mu \sim \epsilon_n \quad (\alpha \uparrow \infty). \tag{1.9}$$

We observe that (i) the significance of (1.9) is limited by the *a priori* restriction $kl_s \ll 1$, (ii) these limiting values may not be discrete, since some of the ϵ_n may be equal, (iii) the ϵ_n, and hence the ${}_n\mu$, may not be ordered by magnitude.

We proceed by posing the eigenvalue problem in the form of a homogeneous Fredholm integral equation (in § 2) and then (in § 3) invoking the standard theory for such equations to obtain a quadratic functional for μ that yields a lower bound to μ_1 for an arbitrary trial function and to μ_n for a suitably restricted trial function. We also obtain an upper bound, but this is useful only for $n = 1$ and small α, say $\alpha < \frac{1}{2}$. A comparison with the known results for a hyperbolic-tangent profile suggests that our results provide an efficient determination of μ_1 and μ_2 for moderate values of α, although the simplest trial functions, e.g. $\exp(-k|y|)$, yield approximations that are only qualitatively correct as $\alpha \uparrow \infty$ for a continuous thermocline. We assume $kd \gg 1$ in §§ 2 and 3 and give the required extension for finite kd in § 4.

† Drazin & Howard (1961) attacked the internal-wave problem (as a special case of the more difficult problem of stratified shear flow) by posing the solution to the differential equation, (2.7) below, in the form $\exp(-k|y|) \times$ (an expansion in powers of λ and α). Their procedure involves only integrals, as opposed to derivatives, of $\rho(y)$ and therefore may be applied to a discontinuous thermocline. It is, however, intrinsically inefficient for non-small α, and the coefficients of higher powers of α become increasingly cumbersome; see, e.g. equation (2.12) below.

The simplest model of a layered thermocline assumes homogeneous layers and yields an essentially algebraic formulation if the number of sheets N, say, is finite. Such a model, which we describe as a *sheeted thermocline*, was adopted by early investigators of internal waves in order to avoid higher transcendental functions; in fact, it now appears at least as realistic as those continuous models that admit exact solutions (see Krauss 1966 for examples). We give two formulations for this model in § 5. The first of these yields a characteristic determinant that has non-zero elements only along its principal and the two adjoining diagonals; on the other hand, it casts λ_N, rather than μ_1, in the role of the dominant eigenvalue and therefore is not well suited to the approximate determination of μ_1 if N is large. The second formulation follows the integral-equation formulation of § 4 and casts μ_1 in the role of the dominant eigenvalue.

The variational approximation proves to be especially efficient for a sheeted thermocline. In particular, the trial function $\exp(-k|y|)$, which is correct only to zero order in the formulation of Drazin & Howard (1961), yields an approximation to μ_1 that coincides with (1.6) as $\alpha \downarrow 0$ and with (1.9) and $\alpha \uparrow \infty$, and is rather accurate for all intermediate α if only the central sheet is significantly stronger than the other sheets in the thermocline.

2. Integral-equation formulation of the eigenvalue problem

We consider internal waves of the form

$$\delta(x,y,t) = \mathscr{R}\{\phi(y)\, e^{ik(x-ct)}\} \tag{2.1}$$

in an unbounded ideal incompressible fluid, where δ is the vertical displacement of a particle, ϕ is its complex amplitude, and x and y are Cartesian coordinates, y being positive upwards. The linearized equations of motion yield (Lamb 1932, § 235)

$$(\rho\phi')' - (g/c^2)\,\rho'\phi - k^2\rho\phi = 0, \tag{2.2}$$

where $\rho(y)$ is the ambient density. The boundary conditions are

$$\phi(-\infty) = \phi(\infty) = 0. \tag{2.3}$$

Both ϕ and the complex amplitude of the perturbation pressure (hydrodynamic plus hydrostatic),

$$\varpi = \rho(c^2\phi' - g\phi), \tag{2.4}$$

must be continuous across a sheet (a discontinuity in ρ).

We now invoke the Boussinesq approximation. By introducing the normalized density $\sigma(y)$ defined by

$$\rho = \tfrac{1}{2}(\rho_+ + \rho_-)(1-\beta\sigma), \quad \beta = (\rho_- - \rho_+)/(\rho_- + \rho_+) \quad [\rho_\pm \equiv \rho(\pm\infty)], \tag{2.5a,b}$$

which implies $\sigma(\pm\infty) = \pm 1$, and the dimensionless parameter

$$\lambda = \beta g/kc^2 \equiv 1/\mu \tag{2.6}$$

and neglecting $\beta\sigma$ relative to 1 in (2.2), we obtain

$$\phi'' + (\lambda k\sigma' - k^2)\,\phi = 0 \quad (\beta \ll 1). \tag{2.7}$$

Similarly, we may replace continuity of ϖ across a sheet by continuity of

$$\psi \equiv \phi' + \lambda k \sigma \phi. \tag{2.8}$$

The error factor implied by the Boussinesq approximation is $1 + O(\beta)$ and is implicit throughout the subsequent development.

Invoking the Green's function for the operator $d^2/dy^2 - k^2$, we transform the eigenvalue problem posed by (2.3) and (2.7) to

$$\phi(y) = \tfrac{1}{2}\lambda \int_{-1}^{1} e^{-k|y-\eta|}\, \phi(\eta)\, d\sigma(\eta) \quad (\beta \ll 1), \tag{2.9}$$

which is a homogeneous Fredholm integral equation with a positive-definite kernel. The theory of such integral equations is classical (Courant & Hilbert 1953, chap. III, § 4) and applies directly to (2.9) if σ (which has a finite domain), rather than y (which has an infinite domain), is regarded as the independent variable. The dominant mode and its eigenvalue may be determined by iteration, but the corresponding determination of higher modes is cumbersome because of the requirement that the trial function for the nth mode be approximately orthogonal to the first $n-1$ modes.

An alternative integration of (2.7) on the hypothesis that

$$\chi^{(\pm)}(y) \equiv e^{\pm ky}\, \phi(y) \sim C_\pm \quad (ky \to \pm\infty), \tag{2.10}$$

where C_+ and C_- are constants, yields the Volterra integral equation

$$\chi^{(\pm)}(y) = 1 \pm \tfrac{1}{2}\lambda \int_{\pm\infty}^{y} \{1 - e^{\mp 2k(\eta - y)}\}\, \chi^{(\pm)}(\eta)\, \sigma'(\eta)\, d\eta, \tag{2.11a}$$

where the upper and lower signs are ordered. By integrating (2.11a) by parts, we obtain the equivalent form

$$\chi^{(\pm)}(y) = 1 - \lambda k \int_{\pm\infty}^{y} e^{\pm 2k\eta}\, d\eta \int_{\pm\infty}^{\eta} e^{\mp 2k\xi}\, \chi^{(\pm)}(\xi)\, \sigma'(\xi)\, d\xi. \tag{2.11b}$$

Requiring the Wronskian of $\chi^{(+)}$ and $\chi^{(-)}$ to vanish yields the eigenvalue equation for λ. Either (2.11a) or (2.11b) may be solved by iteration, starting from the trial solution $\chi^{(\pm)} = 1$. Lighthill (1957) used analogues of (2.11a) and (2.11b) to determine solutions for homogeneous shear flows for large and small k, respectively (but λ does not appear as an eigenvalue in this context). We remark that the hypothesis (2.10) is not uniformly valid as $k \uparrow \infty$ for a continuously stratified flow (for which $\lambda \sim k$), in consequence of which neither (2.11a) nor (2.11b) is useful for such flows if k is large; however, (2.10) *is* uniformly valid for a sheeted thermocline by virtue of the finite asymptotic limits of the eigenvalues (see (1.9)).

Mr Yves Desaubies has carried the solution of (2.11b) through the third iteration and extended Drazin & Howard's result, equation (1.6), to

$$\lambda_1 = 1 + kL + \tfrac{1}{2}k^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1-\sigma^>)(1+\sigma^<) \times \{(\tfrac{1}{2}+\sigma^>)(\tfrac{1}{2}-\sigma^<) - \tfrac{1}{4}\}\, d\eta\, dy + O(\alpha^3), \tag{2.12}$$

where L is given by (1.6b), and in which the arguments of $\sigma^>$ and $\sigma^<$ are, respectively, the larger and smaller of y and η. Higher approximations lead to increasingly cumbersome integrals and are less efficient than the variational approximations of the following section.

3. Variational approximations ($kd \gg 1$)

Multiplying (2.9) through by $\phi(y)$ and integrating with respect to σ over $(-1, 1)$, we obtain

$$\mu = J(\phi, k)/I(\phi) \equiv \check{\mu}(\phi, k), \tag{3.1a}$$

where

$$I(\phi) = \frac{1}{2}\int_{-1}^{1} \phi^2(y)\, d\sigma_y \quad [d\sigma_y \equiv d\sigma(y)] \tag{3.1b}$$

and

$$J(\phi, k) = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} e^{-k|y-\eta|}\phi(y)\,\phi(\eta)\, d\sigma_y\, d\sigma_\eta. \tag{3.1c}$$

The right-hand side of (3.1a) is a minimum with respect to first-order variations of $\phi(y)$ about the true solution of (2.9). Moreover,

$$\mu_1 \geqslant \check{\mu}(\tilde{\phi}, k) \tag{3.2}$$

for any trial function $\tilde{\phi}(y)$ that is bounded and continuous, with equality if and only if $\tilde{\phi} = \phi_1$. [The statements (3.1)–(3.3) follow directly from Courant & Hilbert (1953, chap. III, § 4).]

By invoking the Schwarz inequality

$$J^2(\phi, k) \leqslant J(1, 2k)\, J(\phi^2, 0) \equiv J(1, 2k)\, I^2(\phi), \tag{3.3}$$

we obtain the upper bound $\mu_1^2 \leqslant J(1, 2k)$, with equality if and only if $k = 0$. Combining this result with the lower bound given by $\tilde{\phi} = 1$ and invoking $I(1) = 1$, we obtain upper and lower bounds from a single function:

$$\check{\mu}(1, k) < \mu_1 < \check{\mu}^{\frac{1}{2}}(1, 2k) \equiv \hat{\mu}(k) \quad (k > 0). \tag{3.4}$$

Setting $\tilde{\phi} = 1$ in (3.1c) and integrating by parts, we obtain

$$\check{\mu}(1, k) = 1 - kL + k^2 M(k), \tag{3.5a}$$

where

$$L = \frac{1}{2}\int_{-\infty}^{\infty} (1-\sigma^2)\, dy \equiv L(1.6b), \tag{3.5b}$$

$$M(k) = \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-k|y-\eta|}(1+\sigma^<)(1-\sigma^>)\, d\eta\, dy, \tag{3.5c}$$

and the arguments of $\sigma^>$ and $\sigma^<$ are, respectively, the larger and smaller of y and η. Substituting (3.5a) into (3.4), expanding about $k = 0$, and comparing the results with (1.6), we find that both the upper and the lower bounds are exact to within $O(k^2)$.

The hypothesis (2.10) suggests that the trial function $\exp(-k|y|)$ might be superior to $\tilde{\phi} = 1$. By transforming I and J through integration by parts, we obtain

$$\check{\mu}(e^{-k|y|}, k) = (1-\kappa_1)^{-1}(1-\kappa_1 - \tfrac{1}{2}\kappa_2 + 2\kappa_1^{(+)}\kappa_1^{(-)}), \tag{3.6a}$$

where

$$\kappa_n^{(\pm)} = \pm k\int_0^{\pm\infty} e^{\mp 2ky}\{1-(\pm\sigma)^n\}\, dy \tag{3.6b}$$

and

$$\kappa_n = \kappa_n^{(+)} + \kappa_n^{(-)}. \tag{3.6c}$$

Perhaps the simplest model of a continuous thermocline is provided by

$$\sigma(y) = \tanh(y/L), \tag{3.7}$$

for which L is defined by (3.5*b*), and (2.7) may be transformed to the hypergeometric equation with σ as independent variable (Krauss 1966, pp. 35–38; the model was due originally to Groen). The resulting eigenvalues (for which the hypergeometric series terminate) are

$$\lambda_n = (n-1+\alpha)(n+\alpha)/\alpha \quad (n = 1, 2, \ldots, \infty), \quad \alpha = kL. \tag{3.8a, b}$$

[The algebraic simplicity of the exact result (3.8) *vis-à-vis* the subsequent approximations is atypical and is associated with the rather special properties of (3.7). The approximations are, of course, intended primarily for configurations for which exact solutions are unavailable.] Substituting (3.7) into (3.4)–(3.6), we obtain

$$\check{\mu}(1, k) = 1-\alpha+\tfrac{1}{2}\alpha^2\psi'(\tfrac{1}{2}\alpha+1) \quad (\alpha = kL) \tag{3.9a}$$

$$= 1-\alpha+0{\cdot}823\alpha^2+O(\alpha^3) \quad (\alpha \downarrow 0) \tag{3.9b}$$

$$\sim \tfrac{2}{3}\alpha^{-1}+O(\alpha^{-2}) \quad (\alpha \uparrow \infty), \tag{3.9c}$$

$$\hat{\mu}(k) = \{1-2\alpha+2\alpha^2\psi'(\alpha+1)\}^{\frac{1}{2}} \tag{3.10a}$$

$$= 1-\alpha+1{\cdot}145\alpha^2+O(\alpha^3) \tag{3.10b}$$

$$\sim (3\alpha)^{-\frac{1}{2}}+O(\alpha^{-\frac{3}{2}}), \tag{3.10c}$$

and

$$\check{\mu}(e^{-k|y|}, k) = 1-\alpha+\tfrac{1}{2}(1-\alpha\chi)^{-1}\alpha^2\chi^2 \tag{3.11a}$$

$$= 1-\alpha+0{\cdot}961\alpha^2+O(\alpha^3) \tag{3.11b}$$

$$= \tfrac{3}{4}\alpha^{-1}+O(\alpha^{-2}), \tag{3.11c}$$

where $\psi(z)$ is the logarithmic derivative of $\Gamma(z)$, and

$$\chi(\alpha) = 2\int_0^1 (1+t)^{-1}t^{\alpha}\,dt = \psi'(\tfrac{1}{2}\alpha+1)-\psi'(\tfrac{1}{2}\alpha+\tfrac{1}{2}). \tag{3.12}$$

These approximations are to be compared with the exact result

$$\mu_1 = (1+\alpha)^{-1} \tag{3.13a}$$

$$= 1-\alpha+\alpha^2+O(\alpha^3) \tag{3.13b}$$

$$\sim \alpha^{-1}+O(\alpha^{-2}). \tag{3.13c}$$

Graphical comparisons are given in figure 1. The upper and lower bounds given by (3.9) and (3.10) are less sharp than the lower bound given by (3.11) for all α.† The average of (3.10*b*) and (3.11*b*), $1-\alpha+0{\cdot}984\alpha^2$, is sharper than (3.11*b*) for small α. The lower bounds (3.9*c*) and (3.11*c*) are qualitatively correct, whereas (3.10*c*) gives the wrong order of magnitude, for large α.

† The lower bound (3.5) may be sharper than (3.6) for moderate k if $\sigma(y)$ is less sharply peaked than (3.7); cf. (5.10)–(5.12) below.

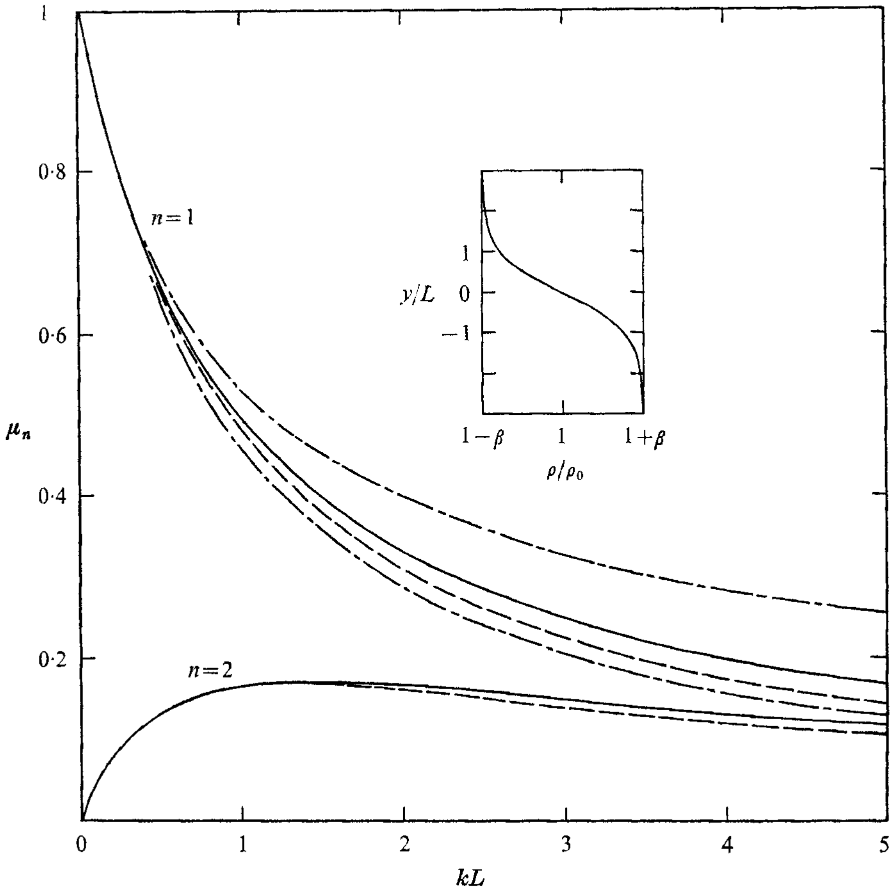


FIGURE 1. The first ($n = 1$) and second ($n = 2$) eigenvalues for the density distribution (3.7), which is sketched in the insert. The exact results, as given by (3.13) and (3.21), are compared with the approximations developed in §3. ——, (3.13) and (3.21); – – –, (3.11) and (3.20); — · —, (3.9) and (3.10).

Asymptotic limits

The counterparts of (3.9*c*), (3.11*c*) and (3.13*c*) for any *continuous* thermocline in which $\sigma'(y)$ has a single peak, σ_0', at $y = 0$ are

$$\check{\mu}(1,k) \sim \tfrac{1}{2}k^{-1}\int_{-\infty}^{\infty} \sigma'^2(y)\,dy, \tag{3.14}$$

$$\check{\mu}(e^{-k|y|},k) \sim \tfrac{3}{4}\sigma_0' k^{-1} \tag{3.15}$$

and

$$\mu_1 \sim \sigma_0' k^{-1}, \tag{3.16}$$

where (3.14) and (3.15) follow from asymptotic approximations to the various integrals and (3.16) follows from Sturm–Liouville theory. The failure of the upper bound of (3.4) to give the correct order of magnitude for large α is connected with the fact that

$$\sum_{n=1}^{\infty} \mu_n^2 = \hat{\mu}^2(k), \tag{3.17}$$

which follows from the bilinear formula for the iterated kernel of (2.9); see Courant & Hilbert (1953, p. 138).

Higher modes

The lower bounds provided by (3.4) and (3.6) may be improved, and approximations to $\mu_2, \ldots, \mu_N$ obtained, by posing a trial function that contains $N-1$ arbitrary parameters, say $A_1, \ldots, A_{N-1}$, which may be determined by minimizing $\breve{\mu}$. The trial functions 1 and $\exp(-k|y|)$ may be generalized through multiplication by an Nth order polynomial in σ if the thermocline is continuous (but such a representation would imply discontinuities in ϕ, and therefore would be inappropriate, for a discontinuous thermocline). The generalization of (3.6) then may be expressed in terms of $\kappa_1^{(\pm)}, \ldots, \kappa_{2N}^{(\pm)}$; in particular,

$$\begin{aligned}\breve{\mu}\{(1+a\sigma)\,e^{-k|y|}, k\} &= \{1-\kappa_1-\kappa_2^* a+\tfrac{1}{3}(1-\kappa_3)\,a^2\}^{-1}\{1-\kappa_1-\tfrac{1}{2}\kappa_2+2\kappa_1^{(+)}\kappa_1^{(-)} \\ &\quad +(\tfrac{1}{2}\kappa_1^*-\tfrac{1}{2}\kappa_2^*-\tfrac{1}{3}\kappa_3^*+\kappa_2^{(+)}\kappa_1^{(-)}-\kappa_2^{(-)}\kappa_1^{(+)})\,a+(\tfrac{1}{4}\kappa_2-\tfrac{1}{8}\kappa_4-\tfrac{1}{2}\kappa_2^{(+)}\kappa_2^{(-)})\,a^2\},\end{aligned} \tag{3.18a}$$

where $\kappa_n^{(\pm)}$ and κ_n are defined by (3.6*b*) and (3.6*c*), and

$$\kappa_n^* = \kappa_n^{(+)}-\kappa_n^{(-)}. \tag{3.18b}$$

The results for a symmetrical thermocline, for which σ is an odd function of y and $\kappa_n^* = 0$, may be simplified by separating the modes into even and odd sets. The trial functions for the first two modes, $\exp(-k|y|)$ and $\sigma\exp(-k|y|)$ (corresponding to $a = 0$ and $a = \infty$, respectively, in (3.18)), yield (3.6) and

$$\breve{\mu}(\sigma\,e^{-k|y|}, k) = \tfrac{3}{8}(1-\kappa_3)^{-1}(2\kappa_2-\kappa_4-\kappa_2^2) \quad (\sigma(y) = -\sigma(-y)). \tag{3.19}$$

Substituting (3.7) into (3.19), we obtain

$$\begin{aligned}\breve{\mu}(\sigma\,e^{-ky}, k) &= \tfrac{1}{2}\alpha\{1-\alpha+2\alpha^2-\alpha(1+2\alpha^2)\,\chi\}^{-1} \\ &\qquad \times\{1-\tfrac{5}{2}\alpha-\alpha^2-\alpha(1-6\alpha-\alpha^2)\,\chi-3\alpha^2\chi^2\} &(3.20a)\\ &= \tfrac{1}{2}\alpha\{1-\tfrac{3}{2}\alpha+1{\cdot}738\alpha^2+O(\alpha^3)\} &(3.20b)\\ &\sim \tfrac{5}{8}\alpha^{-1}+O(\alpha^{-2}), &(3.20c)\end{aligned}$$

where χ is given by (3.12). This is to be compared with the exact result

$$\begin{aligned}\mu_2 &= \alpha(\alpha+1)^{-1}(\alpha+2)^{-1} &(3.21a)\\ &= \tfrac{1}{2}\alpha\{1-\tfrac{3}{2}\alpha+\tfrac{7}{4}\alpha^2+O(\alpha^3)\} &(3.21b)\\ &\sim \alpha^{-1}+O(\alpha^{-2}). &(3.21c)\end{aligned}$$

A graphical comparison is given in figure 1. The agreement between the approximate and exact results is better than that for the dominant mode for moderate α (0·7 % *vs.* 2·5 % error at $\alpha = 1$), but the error in (3.20*c*) is greater than that in (3.11*c*) for large α. The approximation $\breve{\mu}(\sigma, k)$ is found to be in error by 10 % at $\alpha = 1$.

Rayleigh approximation

An alternative variational approximation is provided by the Rayleigh quotient for (2.7) and (2.3):

$$\mu = k\int_{-1}^{1}\phi^2\,d\sigma\Big/\int_{-\infty}^{\infty}(\phi'^2+k^2\phi^2)\,dy. \tag{3.22}$$

However, in contrast to (3.1), which does not involve ϕ', (3.22) requires suitable estimates of both ϕ and ϕ'. In particular, the trial function $\exp(-k|y|)$, which renders ϕ' discontinuous at $y = 0$, yields

$$\mu = \frac{1}{2}\int_{-1}^{1} e^{-2k|y|}\,d\sigma = 1-\kappa_1, \tag{3.23}$$

which differs significantly from (3.6) and is in error by $O(\alpha)$ as $\alpha \downarrow 0$. Substituting (3.7) into (3.23), we find that the error is 21 % at $\alpha = 1$, whereas the corresponding error for (3.11) is only 2·5 %.

4. Thermocline at finite depth

By replacing the boundary condition $\phi(\infty) = 0$ by $\phi(d) = 0$ and modifying the Green's function for the operator $d^2/dy^2 - k^2$ in (2.7) accordingly, we obtain

$$\phi(y) = \tfrac{1}{2}\lambda\int_{-1}^{1} E(y,\eta)\,\phi(\eta)\,d\sigma_\eta, \tag{4.1a}$$

in which

$$E(y,\eta) = e^{-k|y-\eta|} - e^{-k(2d-y-\eta)} \tag{4.1b}$$

replaces the simple exponential kernel in (2.9). Alternatively, we may obtain (4.1) from (2.9) by introducing the image of the original kernel in the surface.

Proceeding as in § 3, we obtain the lower bound (cf. (3.1))

$$\check{\mu}(\phi,k) = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} E(y,\eta)\,\phi(y)\,\phi(\eta)\,d\sigma_y\,d\sigma_\eta/I(\phi) \tag{4.2a}$$

$$= \{J(\phi,k) - K(\phi,k)\}/I(\phi) \tag{4.2b}$$

and the upper bound (cf. (3.3))

$$\hat{\mu}(k) = \frac{1}{2}\left\{\int_{-1}^{1}\int_{-1}^{1} E^2(y,\eta)\,d\sigma_y\,d\sigma_\eta\right\}^{\frac{1}{2}} \tag{4.3a}$$

$$= \{J(1,2k) - 2J(e^{-k(d-y)},k) + K(1,2k)\}^{\frac{1}{2}}, \tag{4.3b}$$

where I and J are defined by (3.1*b*, *c*), and

$$K(\phi,k) = \left\{\frac{1}{2}\int_{-1}^{1} e^{-k(d-y)}\phi(y)\,d\sigma_y\right\}^2. \tag{4.4}$$

The upper bound is *not* exact in the limit $k \downarrow 0$.

The counterparts of (3.5) and (3.6) may be obtained by substituting the respective trial functions $\check{\phi} = 1$ and

$$\check{\phi}(y) = \sinh k(d-y)/\sinh kd \quad (0 < y < d) \tag{4.5a}$$

$$= e^{ky} \quad (y < 0) \tag{4.5b}$$

into (4.2). We give explicit results only for $k \downarrow 0$, in which limit $\phi = 1 + O(kL, L/d)$. Substituting $\phi = 1$ into (4.2*a*), expanding E in powers of k, and integrating by parts, we obtain

$$\mu_1 = \tfrac{1}{4}k\int_{-\infty}^{d}\int_{-\infty}^{d}\{(2d-y-\eta-|y-\eta|) - 2k(d-y)(d-\eta)\}\,\sigma'(y)\,\sigma'(\eta)\,d\eta\,dy + O(k^3) \tag{4.6a}$$

$$= \tfrac{1}{2}k\int_{-\infty}^{d}(1+\sigma)^2\,dy - \tfrac{1}{2}k^2\left\{\int_{-\infty}^{d}(1+\sigma)\,dy\right\}^2 + O(k^3), \tag{4.6b}$$

within an error factor of $1+O(kL^2/d, L^2/d^2)$. Substituting μ from (1.4*b*) and σ from (2.5) into (4.6*b*) and comparing the result to (1.7), we obtain the approximations

$$L_0 = \frac{1}{4}\int_{-\infty}^{d} (1+\sigma)^2\,dy = \int_{-\infty}^{d} \left(\frac{\rho_- - \rho}{\rho_- - \rho_+}\right)^2 dy \tag{4.7a}$$

and
$$L_1 = \left\{\int_{-\infty}^{d} (1+\sigma)\,dy\right\}^2 \Big/ \int_{-\infty}^{d} (1+\sigma)^2\,dy = L_0 + L + \tfrac{1}{4}(L^2/L_0), \tag{4.7b}$$

where L is given by (1.6*b*) with the upper limit of integration therein replaced by d. Both L_0 and L_1 tend to d as $L/d \downarrow 0$.

Those last results, together with the known result for a thin thermocline at finite depth,

$$\mu_1 \to 1 - e^{-2kd} \quad (kL \downarrow 0), \tag{4.8}$$

suggest that the effects of finite depth on the dominant mode may be determined approximately through the relation

$$\mu_1(\alpha, L/d) \doteqdot \mu_1(\alpha, 0)\{1 - \exp(-2kL_0)\} \tag{4.9a}$$

$$= 2kL_0\{1 - k(L+L_0) + O(k^2L_0^2)\}, \tag{4.9b}$$

which implies (1.8).

5. Sheeted thermocline

We now consider a thermocline of homogeneous layers separated by N sheets at $y = y_n$ $(n = 1, 2, \ldots, N)$, each of which is characterized by the density-jump parameter β_n. An implicit solution of (2.2) then is given by

$$\phi(y) = \frac{\phi_{n-1}\sinh k(y_n - y) + \phi_n \sinh k(y - y_{n-1})}{\sinh k(y_n - y_{n-1})} \quad (y_{n-1} \leqslant y \leqslant y_n), \tag{5.1a}$$

where $$\phi_n \equiv \phi(y_n) \quad (y_0 = -\infty, \quad y_{N+1} = d). \tag{5.1b}$$

Invoking continuity of the pressure function ψ (2.8) at $y_1, y_2, \ldots, y_N$ and null conditions at y_0 and y_{N+1} gives

$$\phi_{n-1}\,\mathrm{cosech}\,\kappa_n + (2\epsilon_n\lambda - \coth\kappa_n - \coth\kappa_{n+1})\,\phi_n + \phi_{n+1}\,\mathrm{cosech}\,\kappa_{n+1} = 0 \quad (n = 1, 2, \ldots, N) \tag{5.2a}$$

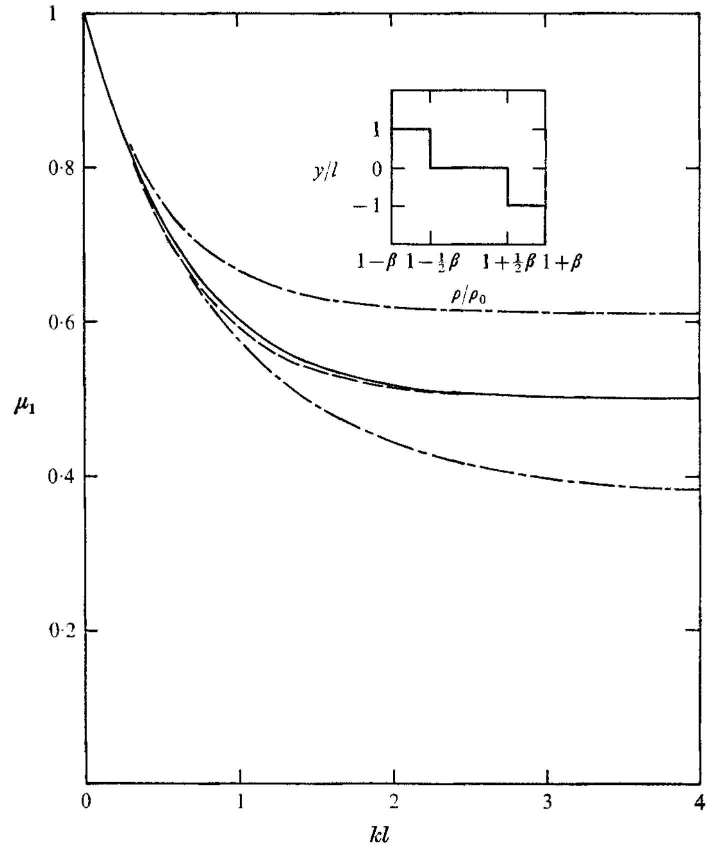
and $$\phi_0 = \phi_{N+1} = 0, \tag{5.2b}$$

where $$\kappa_n = k(y_n - y_{n-1}) \tag{5.3}$$

and ϵ_n is defined by (1.2). The algebraic equations (5.2) are well suited to standard computer routines for moderate N by virtue of the sparseness of their determinant (the only non-zero elements of which lie on the principal and two adjoining diagonals); on the other hand, (5.2) casts λ_N, rather than μ_1, in the role of the dominant eigenvalue.

We obtain a complementary formulation from the integral equation (4.1). Substituting

$$\sigma(y) = 2\sum_{n=1}^{N} \epsilon_n H(y - y_n), \tag{5.4}$$



(*a*)

FIGURE 2. For legend see facing page.

where H is Heaviside's step function, into (4.1) and setting $y = y_m$, we obtain the algebraic equations

$$\phi_m = \lambda \sum_{n=1}^{N} \epsilon_n E_{mn} \phi_n, \quad E_{mn} = E(y_m, y_n), \tag{5.5a, b}$$

or, equivalently,

$$(\mu \mathbf{I} - \mathbf{E}\boldsymbol{\epsilon})\boldsymbol{\phi} = 0, \tag{5.6}$$

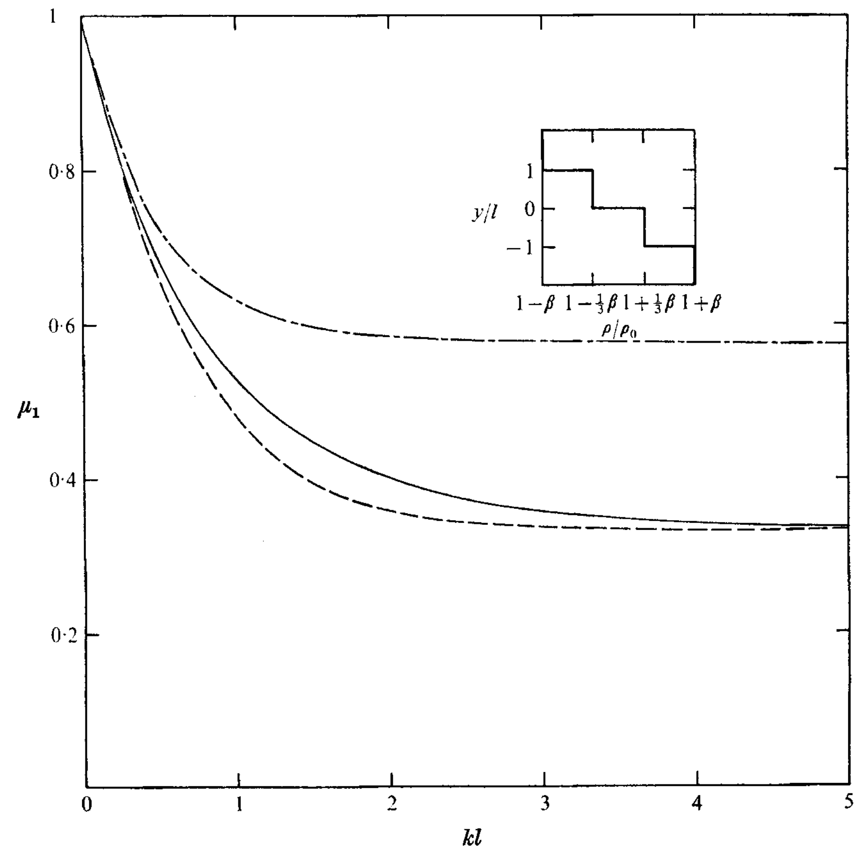
where $\mathbf{I}$ is the unit matrix, $\mathbf{E} = [E_{mn}]$ is a symmetric square matrix, $\boldsymbol{\epsilon} = [\delta_{mn}\epsilon_n]$ is a diagonal matrix, and $\boldsymbol{\phi} = \{\phi_n\}$ is a column matrix. It follows that the μ_n are the latent roots of $\mathbf{E}\boldsymbol{\epsilon}$ and that μ_1 is the dominant root. Standard computer routines are available for the determination of these latent roots, and matrix iteration converges on μ_1.

Turning to the variational approximation, we substitute (5.4) into (3.1 *b*, *c*) and (4.4) to obtain

$$I(\phi, k) = \sum_n \epsilon_n \phi_n^2, \quad J(\phi, k) = \sum_m \sum_n e^{-k|y_m - y_n|} \epsilon_m \epsilon_n \phi_m \phi_n, \tag{5.7a, b}$$

and

$$K(\phi, k) = \{\sum_n e^{-k(d+y_n)} \epsilon_n \phi_n\}^2, \tag{5.7c}$$



(*b*)

FIGURE 2. The dominant-mode eigenvalue for the three-sheeted thermocline described by (5.9). The exact result, as given by (5.10), is compared with the approximations developed in §§3 and 4. ——, (5.10); – – –, (5.12); —·—, (3.4) and (5.11). (*a*) $\epsilon = \frac{1}{2}$. (*b*) $\epsilon = \frac{1}{3}$.

where, here and subsequently, the summations are from 1 to N. Substituting (5.7) into (4.2) and letting $k \uparrow \infty$, we obtain

$$\breve{\mu}(\phi, k) \sim \sum_n \epsilon_n^2 \phi_n^2 / \Sigma \epsilon_n \phi_n^2 \quad (k \uparrow \infty), \tag{5.8}$$

which yields $\mu_n \sim \epsilon_n$, as anticipated in (1.9), for that mode for which ϕ peaks strongly at $y = y_n$.

The results (3.5) and (3.6) remain valid for a sheeted thermocline, but the direct substitution of the trial function $\breve{\phi}$ into (5.7) yields somewhat simpler results. The results (3.18) and (3.19) do not remain valid for a sheeted thermocline, but their counterparts may be obtained by replacing σ by its average value (over $y = y_n \pm$) at $y = y_n$, say $\langle \sigma_n \rangle$.

We illustrate the foregoing results for the three-sheeted thermocline described by

$$\epsilon_n = \tfrac{1}{2}(1-\epsilon), \epsilon, \tfrac{1}{2}(1-\epsilon) \quad \text{at} \quad y_n = -l, 0, l \quad (n = 1, 2, 3) \tag{5.9}$$

and $y_4 = \infty$ ($kd \gg 1$). Substituting these data into either (5.2) or (5.6) and requiring the determinant of the equations in $\phi_{1,2,3}$ to vanish, we obtain

$$\mu_{1,3} = \tfrac{1}{2}[1-\mu_2 \pm \{(1-\mu_2)^2 - 4\epsilon\mu_2\}^{\frac{1}{2}}] \tag{5.10a}$$

and

$$\mu_2 = \tfrac{1}{2}(1-\epsilon)(1-e^{-2kl}), \tag{5.10b}$$

where the numbering of the eigenvalues is such that $\mu_1 > \mu_2 > \mu_3$ and has no direct correspondence with the numbering of the layers. The dominant eigenvalue μ_1 is plotted in figures 2(*a*) and (*b*) for $\epsilon = \frac{1}{2}$ and $\frac{1}{3}$. The dimensionless dispersion curves, $(\beta g/L)^{-\frac{1}{2}}\,\omega$ *vs.* $\alpha = kL = kl(1-\epsilon^2)$, are plotted in figure 3 for several values of ϵ in order to illustrate the sensitivity of the results to ϵ for moderate values of α when L is regarded as the characteristic length. The dispersion curve for $\epsilon = \frac{1}{3}$ (sheets of equal strength) differs from that for the hyperbolic-tangent model by less than 5% for $\alpha \leqslant 1$.

By substituting (5.9) into (5.7*a*, *b*) and invoking the trial functions 1 and $\exp(-k|y|)$, we obtain the variational approximations

$$\check{\mu}(1,k) = \epsilon^2 + \tfrac{1}{2}(1-\epsilon)^2 + 2\epsilon(1-\epsilon)\,e^{-kl} + \tfrac{1}{2}(1-\epsilon)^2\,e^{-2kl} \tag{5.11}$$

and

$$\check{\mu}(e^{-k|y|},k) = \frac{\epsilon^2 + \frac{1}{2}(1-\epsilon)(1+3\epsilon)\,e^{-2kl} + \frac{1}{2}(1-\epsilon)^2\,e^{-4kl}}{\epsilon + (1-\epsilon)\,e^{-2kl}}, \tag{5.12}$$

which are compared with the exact result (5.10) in figure 2 for $\epsilon = \frac{1}{2}$ and $\epsilon = \frac{1}{3}$. The approximation (5.12) is scarcely distinguishable from the exact result (on the scale of figure 2) for $\frac{1}{2} < \epsilon \leqslant 1$; on the other hand, the approximation is less satisfactory (except for small kl) for $0 < \epsilon < \frac{1}{3}$, in which range the outer sheets are stronger than the central sheet. The lower bound provided by (5.11) is inferior to that of (5.12) if ϵ is appreciably greater than $\frac{1}{3}$ but is superior if $\epsilon \leqslant \frac{1}{3}$; indeed, it is indistinguishable (on the scale of figure 2) from the exact result for $\epsilon = \frac{1}{3}$, presumably because it is asymptotically exact in this special case. The upper bound provided by (5.11) in conjunction with (3.4) is sharp only for small k. Invoking the trial function $\phi_n = \langle\sigma_n\rangle \exp(-k|y_n|)$ yields the exact result (5.10*b*) for the second mode and does not provide a test of the variational approximation.

As a final example, we consider the five-sheeted thermocline described by

$$\epsilon_n = \tfrac{1}{8}, \tfrac{3}{16}, \tfrac{3}{8}, \tfrac{3}{16}, \tfrac{1}{8} \quad \text{at} \quad y_n/l = -3, -1, 0, 1, 3,$$
$$d/l = 5 \quad (n = 1, 2, 3, 4, 5), \tag{5.13}$$

which approximates the summer thermocline in the Mediterranean (Woods 1968, figure 2) with $l = 5$ m and $\beta = 1{\cdot}1 \times 10^{-3}$ (using a coefficient of thermal expansion for sea water of $2{\cdot}5 \times 10^{-4}$/°C); see figure 4, in which the bars represent the observed sheets. Substituting (5.13) into (5.4), (1.6*b*), and (4.7*a*, *b*), we obtain $(L, L_0, L_1) = (1{\cdot}73, 4{\cdot}13, 6{\cdot}05)\,l$. The first two eigenvalues, as obtained from (5.6), are plotted in figure 5 for $d = 5l$ and $d = \infty$ in order to illustrate the effects of the free surface. These effects are qualitative for the dominant mode and only quantitative for the higher modes. The approximation (4.9*a*) cannot be distinguished from the exact result on the scale of figure 5 and exhibits a maximum error of 2% at $kl = 0{\cdot}1$. The dispersion curves for the five modes are plotted in figure 6; the asymptotic coalescence of the curves for the second and third modes

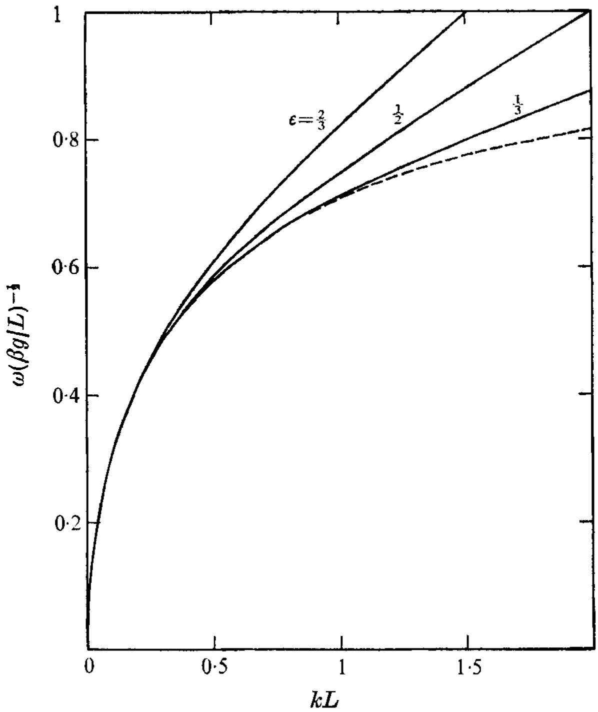


FIGURE 3. The dispersion curves. ——, three-sheeted thermocline described by (5.9); – – –, continuous thermocline described by (3.7).

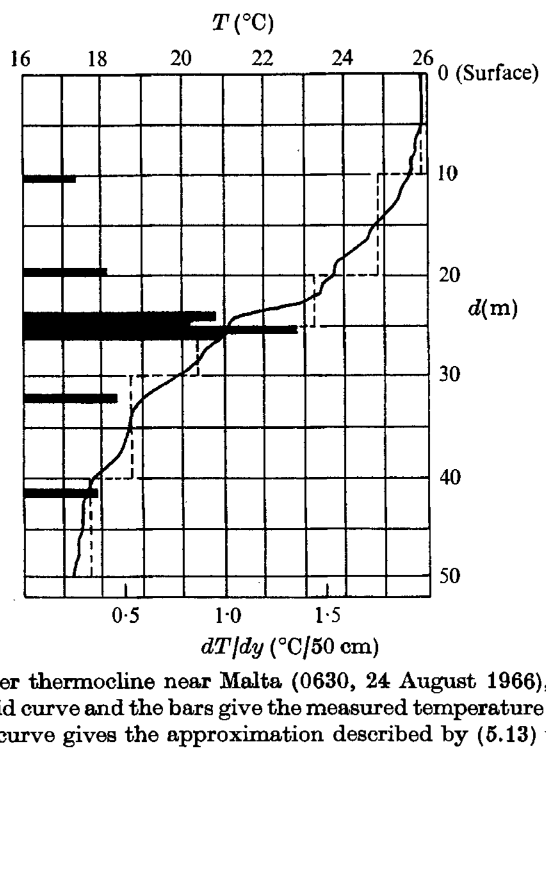


FIGURE 4. The summer thermocline near Malta (0630, 24 August 1966), as reported by Woods (1968). The solid curve and the bars give the measured temperature and temperature gradient. The dashed curve gives the approximation described by (5.13) with $l = 5\,\text{m}$ and $\beta = 1{\cdot}1 \times 10^{-3}$.

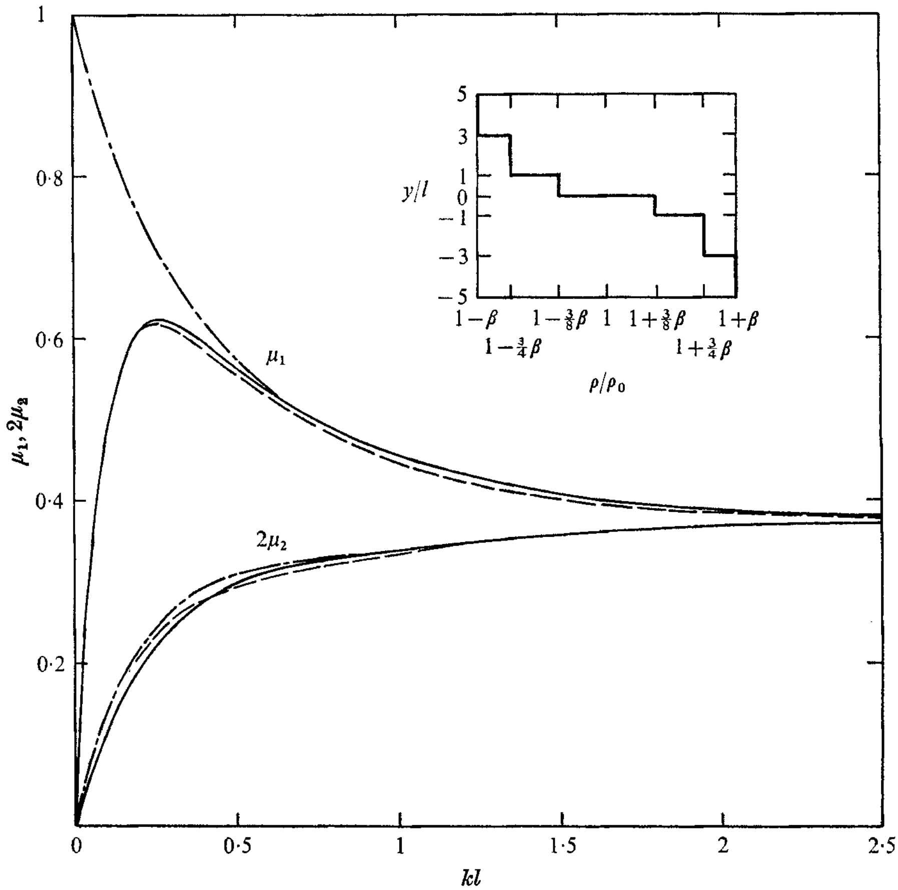


FIGURE 5. The first two eigenvalues for the five-sheeted mode of the thermocline described by (5.13). ——, $d = 5l$; — · —, $d = \infty$; – – –, variational approximations.

and those for the fourth and fifth modes is a consequence of the symmetry of the thermocline with respect to its midplane.

The variational approximations to μ_1 and μ_2, as obtained by substituting the trial functions (4.5) and $\langle\sigma\rangle_n \exp(-k|y_n|)$, respectively, into (5.4) and (5.7) are also plotted in figure 5. The second mode is strictly antisymmetric only for $kd \gg 1$, in consequence of which the variational approximation to μ_2 does not provide a lower bound to the exact value for small kl. The upper bound to μ_1 provided by (4.3) is comparable in accuracy with the corresponding bound in figure 2(*a*).

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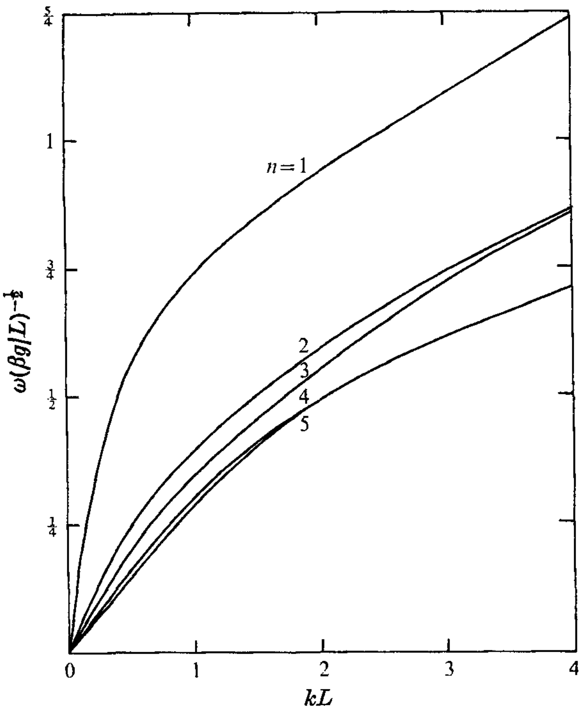

FIGURE 6. The dispersion curves for the five-sheeted thermocline described by (5.13).

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